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J. Math. Anal. Appl. 269 (2002) 616–641

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

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Optimal control of causal differential–algebraic systems ☆

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Received 19 July 2000; received in revised form 24 September 2001; accepted 30 November 2001

Submitted by L. Berkovitz

Abstract

Existence theory of Filippov–Roxin type as well as a maximum principle for optimal control problem governed by nonlinear differential–algebraic equations of index 1, or 2, or 3 are formulated and proved. The index-3 case is illustrated on mechanical descriptor systems arising in robotics. © 2002 Elsevier Science (USA). All rights reserved.

Keywords: Optimal control; Differential–algebraic equations; Existence theory; Maximum principle; Orientor field; Robots

☆ The authors are thankful to Ondřej Vlček for reading and improving earlier versions of the paper; in particular Remark 5 is due to him. Also, the authors are indebted to the anonymous referee(s) for valuable comments that lead to improvements of the paper in many places. This research was partly covered by the grants MSM 11320007 and J04/98:212200003 (MŠMT ČR), 201/00/0768 and 101/99/0729 (GA ČR), and A 107 5005 (GA AV ČR).

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PII: S0022-247X(02)00040-9

1. Introduction

This paper is focused on differential–algebraic equations (abbreviated as DAEs, as usual) in the semi-explicit form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x, y, u), \quad x(0) = x_0, \tag{1.1a}$$

$$0 = g(t, x, y, u) \tag{1.1b}$$

for $x(t) \in \mathbb{R}^{n_1}$ and $y(t) \in \mathbb{R}^{n_2}$ unknown vectors of "slow" and "fast" variables, respectively, and $u(t) \in \mathbb{R}^m$ the vector of parameters, later used as control. Thus $f : (0, T) \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_1}$ and $g : (0, T) \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_2}$ with $n_1, n_2, m \in N$. Saying that (1.1) has (differential) *index* k means that we need to differentiate the algebraic part (1.1b) $(k - 1)$-times in time to get the underlying system of ODE; cf. [1,2].

Sometimes, DAE may exhibit hidden effects related with $\mathrm{d}^i u/\mathrm{d}t^i$, $i \geqslant 1$. As usual (see, e.g., [3,4]), we call DAE *causal* if the solution $[x, y](t)$ does not depend on the derivatives $\mathrm{d}u/\mathrm{d}t, \ldots, \mathrm{d}^{(k-1)}u/\mathrm{d}t^{(k-1)}$ at a current time t but on $u(t)$ only.

Our aim is to derive a maximum principle as well as a Filippov–Roxin-type convexity condition for the existence of an optimal control of Bolza's problem involving causal DAE (1.1) of index $k > 1$ (Sections 5 and 6). This does not seem to be known so far. Even in the case $k = 1$ (Section 4), our results extend or improve the known results; see [5] where only convex orientor fields are admitted for the maximum principle (while a local optimality conditions, called a weak maximum principle in [5], are derived in a nonconvex case) or [6] which does not specify any equation for the fast adjoint variable μ in contrast to our results—see (2.3b) or (5.8b) below. The importance of this equation is both for selectivity of optimality conditions (cf. Section 2) and for the formula (4.23) leading, in a convex case, to a maximal principle involving controls only. To be more specific, let us also mention that both [5] and [6] admit nonsmooth problems which, however, we do not deal with. Besides, in case $k = 3$ we apply our results to mechanical descriptor systems arising in robotics (Section 7), which is illustrative and again seems to be a new achievement. For clarity of explanation, we confine ourselves to systems which have the same time-independent index in all equations. Combinations of equations with various time-independent indices are possible, as well as generalizations to systems with a higher index.

Our strategy will rely simply on finding a suitable transformation of the controlled system of DAEs to usual optimal control problem for underlying ordinary differential equations (= ODEs). Supposing we have some data qualification which guarantees the validity of, say, maximum principle and existence theory for the latter problem (see Section 3), the corresponding inverse transformation then yields the data qualification, the maximum principle and existence theory for the original problem with DAEs. The key moment resulting from this transformation

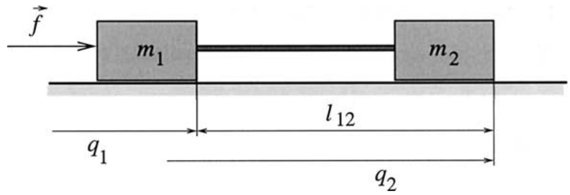

Fig. 1. Two connected moving mass particles: an example of an index-3 system.

is the identification of a manifold $\mathcal{M}$ of the admissible pairs of (y, u), taking into account the phenomenon typical for DAEs, namely that the fast variables can "oscillate" as fast as controls. This manifold, whose form depends on the index of the system of DAEs in question, appears in the maximum principle and incidentally also in the existence condition of the Filippov–Roxin type. Assuming additionally convexity in terms of u, the maximization over the manifold $\mathcal{M}$ can be replaced, in a more standard way, by a maximization over u's but with a modified Hamiltonian. Let us remark that, in some cases, the maximum principle can be even used for synthesis of the optimal control; cf. [7] for the context of robotics.

As an example of a causal DAE system, let us consider the following, rather academic but hopefully illustrative problem; see Section 7 for a general version or Section 2 for another causal DAE. Let two mass particles with masses m_1, m_2, respectively, move on a plane without friction under the action of the force $\vec{f}$ connected by a firm, rigid rod keeping the fixed distance l_{12} (see Fig. 1).

Let us denote the horizontal positions of the particles q_1, q_2, respectively. Obviously, $q_2 - q_1 = l_{12}$. This system is described by standard Lagrange equations of mixed type (cf. [8])

$$m_1 \frac{\mathrm{d}^2 q_1}{\mathrm{d}t^2} = \vec{f} - w, \tag{1.2a}$$

$$m_2 \frac{\mathrm{d}^2 q_2}{\mathrm{d}t^2} = w, \tag{1.2b}$$

$$q_2 - q_1 - l_{12} = 0. \tag{1.2c}$$

Here the mechanical Lagrange multiplier w expresses the interaction force in the connecting rod. These equations obtain the form (1.1) with $f = (x_2, (u - y)/m_1, x_4, y/m_2)$ and $g = (x_3 - x_1 - l_{12})$ after designation $x := (q_1, (\mathrm{d}/\mathrm{d}t)q_1, q_2, (\mathrm{d}/\mathrm{d}t)q_2)$, $y := w$ and $u := \vec{f}$. In order to solve it for $y := w$, it is necessary to differentiate Eq. (1.2c) twice with respect to time and substitute Eqs. (1.2a), (1.2b), thus obtaining $w = m_2 \vec{f}/(m_1 + m_2)$. This shows that Eqs. (1.2) are of index 3.

On the other hand, if one modifies the system (1.2) by adding a control $\vec{g}$ directly into the algebraic constraint (1.2c), i.e.,

$$q_2 - q_1 - l_{12} - \vec{g} = 0, \tag{1.2c$'$}$$

and transforms the modified system (1.2a), (1.2b), (1.2c′) into the form (1.1) by designating $u := (\vec{f}, \vec{g})$, one gets a system which is not causal because, after solving it for $w = m_2(\vec{f} + m_1(\mathrm{d}^2/\mathrm{d}t^2)\vec{g})/(m_1 + m_2)$, the resulting differential equations are

$$\frac{\mathrm{d}^2 q_1}{\mathrm{d}t^2} = \frac{\vec{f}}{m_1 + m_2} - \frac{m_2}{m_1 + m_2}\frac{\mathrm{d}^2 \vec{g}}{\mathrm{d}t^2}, \tag{1.3a}$$

$$\frac{\mathrm{d}^2 q_2}{\mathrm{d}t^2} = \frac{\vec{f}}{m_1 + m_2} + \frac{m_1}{m_1 + m_2}\frac{\mathrm{d}^2 \vec{g}}{\mathrm{d}t^2}, \tag{1.3b}$$

and the solution depends on the time derivatives of the control $\vec{g}$. For other examples of noncausal systems see [4]. The physical interpretation of causality of DAE based on multibody mechanical systems (as (1.2) or (7.1) below) is that the parameters (= the control) $u(t)$ influence the system through the right-hand side of the differential equation (1.1a) and not through the algebraic part (1.1b). The algebraic part in multibody mechanics poses the kinematic description of the positions of bodies. The influence of the parameters (control) on a mechanical system through the algebraic part (1.1b) in multibody mechanics means that their effect is immediate in time. This means immediate change of system momenta and energy, which further means infinitely large acting forces, which is impossible. The influence through differential equations represents the evolution of system momenta and energy during a finite time interval, therefore by finite forces. Besides, the forces acting on the mechanical system enter Eq. (1.1) on the right-hand side of the differential equations (1.1a). (See also Eqs. (7.1).) Thus the vector of parameters $u(t)$ has the natural interpretation as the controlling applied forces.

2. Problem formulation

We will deal with the following optimal control problem in the Bolza form for the (semi-explicit) system of DAEs:

$$(\mathsf{P})\quad \begin{cases} \text{Minimize } I(x,y,u) := \int_0^T h(t,x,y,u)\,\mathrm{d}t + l(x(T)), & \text{(cost functional)} \\ \text{subject to } \frac{\mathrm{d}x}{\mathrm{d}t} = f(t,x,y,u) \text{ on } (0,T), & \text{(differential system)} \\ \qquad 0 = g(t,x,y,u) \text{ on } (0,T), & \text{(algebraic system)} \\ \qquad x(0) = x_0, & \text{(initial condition)} \\ \qquad u(t) \in U(t) \text{ for a.a. } t \in (0,T), & \text{(control constraints)} \\ \qquad u \in L^p(0,T;\mathbb{R}^m), \quad x \in W^{1,p_1}(0,T;\mathbb{R}^{n_1}), & \\ \qquad y \in L^{p_2}(0,T;\mathbb{R}^{n_2}), & \end{cases}$$

where x is the so-called slow variable while y is called fast variable, the time horizon T is fixed. As to the data, we consider $h:(0,T)\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^m\to\mathbb{R}$, $f:(0,T)\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^m\to\mathbb{R}^{n_1}$, and $g:(0,T)\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^m\to\mathbb{R}^{n_2}$

Carathéodory functions (i.e., measurable with respect to t and continuous with respect to the remaining variables), $l:\mathbb{R}^{n_1}\to\mathbb{R}$ continuous, $x_0\in\mathbb{R}^{n_1}$, and $U:(0,T)\rightrightarrows\mathbb{R}^m$ a measurable set-valued mapping with nonempty closed values ("measurable" means, as usual, that all sets $\{t\in(0,T);\ U(t)\subset A\}$, with $A\subset\mathbb{R}^m$ open, are measurable), $n_1,n_2,m\in\mathbb{N}$, $p\in[1,+\infty)$, $p_1,p_2\in(1,+\infty)$. As standard, L^p denotes a Lebesgue space of measurable functions whose p-power is integrable, and $W^{1,p}$ stands for a Sobolev space of functions whose distributional derivative lives in L^p.

There is a quite common belief in literature [4,9–11] that one can apply the standard maximum principle to causal DAE as usual, which would lead in the case of our problem (P) to the maximum principle

$$\mathcal{H}_{x,y,\lambda,\mu}\big(t,u(t)\big)=\max_{u\in U(t)}\mathcal{H}_{x,y,\lambda,\mu}(t,u)\tag{2.1}$$

for a.a. $t\in(0,T)$, where the Hamiltonian $\mathcal{H}_{x,y,\lambda,\mu}:(0,T)\times\mathbb{R}^m\to\mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{H}_{x,y,\lambda,\mu}(t,u):={}&\lambda(t)^{\mathrm{T}}\cdot f\big(t,x(t),y(t),u\big)+\mu(t)^{\mathrm{T}}\cdot g\big(t,x(t),y(t),u\big)\\&-h\big(t,x(t),y(t),u\big)\end{aligned}\tag{2.2}$$

with the superscript "T" denoting the transposition and with x and y solving the DAEs (1.1) and λ and μ being a so-called adjoint state governed by the "adjoint" DAEs

$$\begin{aligned}&\frac{\mathrm{d}\lambda}{\mathrm{d}t}=\frac{\partial h}{\partial x}(t,x,y,u)^{\mathrm{T}}-\frac{\partial f}{\partial x}(t,x,y,u)^{\mathrm{T}}\lambda-\frac{\partial g}{\partial x}(t,x,y,u)^{\mathrm{T}}\mu,\\&\quad\lambda(T)=\frac{\mathrm{d}l}{\mathrm{d}x}\big(x(T)\big),\end{aligned}\tag{2.3a}$$

$$0=\frac{\partial h}{\partial y}(t,x,y,u)^{\mathrm{T}}-\frac{\partial f}{\partial y}(t,x,y,u)^{\mathrm{T}}\lambda-\frac{\partial g}{\partial y}(t,x,y,u)^{\mathrm{T}}\mu;\tag{2.3b}$$

here we used the convention accepted thorough the paper that $x\in\mathbb{R}^{n_1}$, $y\in\mathbb{R}^{n_2}$ and $u\in\mathbb{R}^m$ denotes the variables where $x(t)$, $y(t)$ and $u(t)$ is to be substituted, respectively.

However, the following simple index-1 example shows that (2.1)–(2.3) need not hold for an optimal solution (u,x,y) to (P). Taking $n_1=n_2=m=1$ and a parameter $\alpha\in\mathbb{R}$, we will consider the following problem

$$\begin{cases}\text{Minimize }\int_0^T x^2+\alpha(y-u)^2\,\mathrm{d}t,\\ \text{subject to }\frac{\mathrm{d}x}{\mathrm{d}t}=y-u,\quad x(0)=0,\\ \qquad 0=y-u\text{ on }(0,T),\\ \qquad u(t)\in[-1,1]\text{ for a.a. }t\in(0,T),\\ \qquad u\in L^\infty(0,T),\quad x\in W^{1,\infty}(0,T),\quad y\in L^\infty(0,T).\end{cases}\tag{2.4}$$

Obviously, the minimum in (P), being equal to 0, is attained on every admissible control. In particular, the control $u = 0$ is optimal. Then obviously $x = y = 0$ and the adjoint system (2.3), which now looks as

$$\frac{d\lambda}{dt} = 2x, \quad \lambda(T) = 0, \tag{2.5a}$$

$$0 = 2\alpha(y - u) - \lambda - \mu, \tag{2.5b}$$

gives $\lambda = \mu = 0$. Therefore, the Hamiltonian (2.2) becomes

$$\mathcal{H}_{x,y,\lambda,\mu}(t, u) = -x^2(t) - \alpha\big(y(t) - u\big)^2. \tag{2.6}$$

Yet, if $\alpha < 0$, the maximum of the Hamiltonian (2.6) on $[-1, 1]$ is attained at $u = 1$ or $u = -1$ because $y(t) = 0$, indicating thus that $u = 0$ cannot be an optimal control. This is a simple counterexample to [4, Theorem 5]. After a slight modification, namely considering $n_1 = 2$ and $dx/dt = (y - u, x_1^2 + \alpha(y - u)^2)$, the cost functional in the Mayer form with $h = 0$ and $l(x) = x_2$, this example gives also a counterexample to [9, Theorem 3.1] and [11, Theorem 4.1]; note that $\partial g/\partial y$ does not depend on (x, y, u) as assumed in [9]. For such a Mayer problem, De Pinho and Vinter [5] constructed counterexample that is essentially (2.4) without x-variable. Notice that, however, the counterexample (2.4) does not work if $\alpha \geqslant 0$, which we will explain in Remark 3.

The maximum principle (2.1)–(2.3) was rigorously proved by De Pinho and Vinter [5] for Mayer problems with index-1 systems and convex orientor field. Our modification of (2.1)–(2.3) (see (4.9), (4.10), (6.9), (6.10) below), allows us to eliminate the restrictions on the index and the convexity.

Remark 1. The algebraic part (1.1b) could be, in principle, treated as a state constraint in (P). Yet, though this may give formally the similar results, such an approach would neglect the special character of (1.1b) which does not impose any restriction on the control u, would yield a different optimality conditions involving an additional scalar multiplier (possibly being equal 0), and would bring technical troubles with failure of continuity into L^∞-type space usually required. For example, for index-1 Mayer-type (even nonsmooth) problems, Devdariani and Ledyaev [6] essentially proposed to use such approach for DAEs; i.e., to consider (y, u) as a new control and (1.1b) as a control/state constraint but then [6, Corollary on pp. 82–83] even does not yield any specification (2.3b) of the fast adjoint variable μ except its mere existence, which, however, deteriorates selectivity of resulting optimality conditions. Indeed, even the slow adjoint variable λ is not specified by (2.3a) if $\partial g/\partial x$ is surjective and μ can be arbitrary. For example, in the above example by omitting (2.5b) one gets the maximum principle for the Hamiltonian $\mathcal{H}_{x,y,\lambda,\mu}(t, u) = \mu(t)(y(t) - u) - x^2(t) - \alpha(y(t) - u)^2$ with nonspecified $\mu(t)$, which has no selectivity at all. Moreover, Eq. (2.3b) permits the formulation of alternative versions of the maximum principle in the convex case (see Remark 3 below).

3. Preliminaries: Optimal control of ODEs

Let us summarize briefly some results concerning a standard auxiliary Bolza optimal control problem for a system of ODE:

$$(\mathsf{AP})\quad \begin{cases} \text{Minimize } I(x,u) := \int_0^T \psi(t,x,u)\,\mathrm{d}t + l(x(T)), & \text{(cost functional)} \\ \text{subject to } \frac{\mathrm{d}x}{\mathrm{d}t} = \varphi(t,x,u) \text{ on } (0,T), & \text{(differential system)} \\ \qquad x(0) = x_0, & \text{(initial condition)} \\ \qquad u(t) \in U(t) \text{ for a.a. } t \in (0,T), & \text{(control constraints)} \\ \qquad u \in L^p(0,T;\mathbb{R}^m), \quad x \in W^{1,p_1}(0,T;\mathbb{R}^n). \end{cases}$$

We assume the following growth, Lipschitz-continuity, and coercivity qualifications concerning the Carathéodory mappings $\varphi : (0,T) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\psi : (0,T) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ (cf. [12, Section 4.3a]):

$$\big|\varphi(t,x,u)\big| \leqslant \big(a_{p_1}(t) + c|u|^{p/p_1}\big)(1+|x|), \tag{3.1a}$$

$$\big|\varphi(t,x,u) - \varphi(t,\tilde{x},u)\big| \leqslant \big(a_1(t) + \beta\big(|x| + |\tilde{x}|\big) + c|u|^p\big)|x - \tilde{x}|, \tag{3.1b}$$

$$\varepsilon|u|^p \leqslant \psi(t,x,u) \leqslant a_1(t) + \beta(|x|) + c|u|^p, \tag{3.1c}$$

$$\big|\psi(t,x,u) - \psi(t,\tilde{x},u)\big| \leqslant \big(a_1(t) + \beta\big(|x| + |\tilde{x}|\big) + c|u|^p\big)|x - \tilde{x}|, \tag{3.1d}$$

$$l(x) \geqslant 0, \tag{3.1e}$$

where $x, \tilde{x} \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $a_1 \in L^1(0,T)$, $a_{p_1} \in L^{p_1}(0,T)$, $b, c \in \mathbb{R}$, and $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ arbitrary continuous increasing. The set-valued mapping $U : (0,T) \rightrightarrows \mathbb{R}^m$ was already qualified in Section 2. The following assertion gives only one of possible nontrivial tools to ensure existence of solutions to (AP). It deals with the so-called orientor field Q is defined by

$$Q(t,x) := \big\{(a,b) \in \mathbb{R} \times \mathbb{R}^n;\ a \geqslant \psi(t,x,u),\\ b = \varphi(t,x,u),\ u \in U(t)\big\}. \tag{3.2}$$

Theorem 1 (Filippov [13], Roxin [14], here modified). *Let* (3.1) *be valid and* $Q(t,x)$ *be convex for all* $x \in \mathbb{R}^n$ *and a.a.* $t \in (0,T)$. *Then* (AP) *has a solution.*

For the particular case (3.1), the proof can be done by combining the results from [12, Section 4.3.b] and [15, Lemmas 1 and 2], realizing also that the convexity of Q is quite obviously (cf., e.g., [16, formula (3.1)]) equivalent with

$$\mathrm{co}[\psi \times \varphi]\big(t,x,U(t)\big) \subset Q(t,x) \tag{3.3}$$

for all $x \in \mathbb{R}^n$ and a.a. $t \in (0,T)$.

Furthermore, the following assertion gives a fairly general version of the maximum principle, developed especially by Hestenes [17] and Boltyanskiĭ et al. [18] even for state-constrained problems; cf. Boltyanskiĭ [19] for an interesting historical survey. To derive it by conventional smooth technique for a quite general situation with U possibly unbounded, we need additionally the following smoothness hypotheses on $\psi(t,\cdot,u)$ and $\varphi(t,\cdot,u)$ (cf. [12, Section 4.3a]) and on l:

$$\left|\frac{\partial\varphi}{\partial x}(t,x,u)\right| \leqslant a_{p_1}(t)+\beta(|x|)+c|u|^{p/p_1}, \tag{3.4a}$$

$$\left|\frac{\partial\varphi}{\partial x}(t,x,u)-\frac{\partial\varphi}{\partial x}(t,\tilde{x},u)\right| \leqslant \big(a_{p_1}(t)+\beta\big(|x|+|\tilde{x}|\big)+c|u|^{p/p_1}\big)|x-\tilde{x}|, \tag{3.4b}$$

$$\left|\frac{\partial\psi}{\partial x}(t,x,u)\right| \leqslant a_1(t)+\beta(|x|)+c|u|^{p}, \tag{3.4c}$$

$$\left|\frac{\partial\psi}{\partial x}(t,x,u)-\frac{\partial\psi}{\partial x}(t,\tilde{x},u)\right| \leqslant \big(a_1(t)+\beta\big(|x|+|\tilde{x}|\big)+c|u|^{p}\big)|x-\tilde{x}|, \tag{3.4d}$$

$$l \text{ continuously differentiable.} \tag{3.4e}$$

Theorem 2 (Maximum principle; Boltyanskiĭ et al. [18], Hestenes [17], here modified). *Let* (3.1) *and* (3.4) *hold and let* (u,x) *solve* (AP). *Then the maximum principle*

$$\mathcal{H}_{x,\lambda}\big(t,u(t)\big)=\max_{\tilde{u}\in U(t)}\mathcal{H}_{x,\lambda}(t,\tilde{u}) \tag{3.5}$$

holds for a.a. $t\in(0,T)$, *where the Hamiltonian* $\mathcal{H}_{x,\lambda}:(0,T)\times\mathbb{R}^m\to\mathbb{R}$ *is defined by*

$$\mathcal{H}_{x,\lambda}(t,u):=\lambda(t)^{\mathrm{T}}\cdot\varphi\big(t,x(t),u\big)-\psi\big(t,x(t),u\big), \tag{3.6}$$

where $x\in W^{1,p_1}(0,T;\mathbb{R}^n)$ *solves the initial-value problem in* (AP) *and the adjoint state* $\lambda\in W^{1,1}(0,T;\mathbb{R}^n)$ *satisfies the adjoint terminal-value problem*

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t}=\left[\frac{\partial\psi}{\partial x}(t,x,u)\right]^{\mathrm{T}}-\left[\frac{\partial\varphi}{\partial x}\right]^{\mathrm{T}}(t,x,u)\lambda, \quad \lambda(T)=\frac{\mathrm{d}l}{\mathrm{d}x}\big(x(T)\big). \tag{3.7}$$

The maximum principle (3.5), in fact, represents the standard first-order condition for minimizing a smooth functional on a convex set, which arises by a suitable extension of the problem (AP); we refer to [12, Section 4.3.c] for many technical details.

Let us also mention that the alternative condition (3.3) makes a basis for various nonconvex refinements of Theorem 1 in particular cases by a detailed analysis of the maximum principle (3.5), namely by replacing $U(t)$ in (3.3) by (an upper estimate of) the set where $\mathcal{H}_{x,\lambda}(t,\cdot)$ is maximized; cf. Gabasov et al. [20–22] or also [16].

4. Index-1 problems

The above example indicates that a certain modification of standard results is ultimately needed. We will demonstrate it first on the simplest DAEs with index 1, assuming that the algebraic part (1.1b) admits an implicit function F in the sense

$$\begin{aligned}&\exists F:(0,T)\times\mathbb{R}^{n_1}\times\mathbb{R}^m\to\mathbb{R}^{n_2}:\\ &\quad g(t,x,y,u)=0 \quad\Leftrightarrow\quad y=F(t,x,u).\end{aligned} \tag{4.1}$$

This assumption is to be verified in each particular case in concrete applications; cf. (7.7), (7.8) below for illustration. Furthermore, the following growth condition on F will be useful:

$$\left|F(t,x,u)\right|\leqslant a_{p_2}(t)+\beta(|x|)+c|u|^{p/p_2} \tag{4.2}$$

with some $a_{p_2}\in L^{p_2}(0,T)$, $\beta:\mathbb{R}^+\to\mathbb{R}^+$ arbitrary continuous increasing, and $c\in\mathbb{R}$. The assumption (4.1) allows us to transform the problem (P) into the form of (AP):

Lemma 1. *Let* (4.1) *and* (4.2) *hold. Then the original problem* (P) *is equivalent with the auxiliary problem* (AP) *with*

$$\begin{aligned}&\varphi(t,x,u):=f\big(t,x,F(t,x,u),u\big)\\ &\textit{and}\quad \psi(t,x,u):=h\big(t,x,F(t,x,u),u\big)\end{aligned} \tag{4.3}$$

in the sense that

$$(u,x)\in\mathrm{Argmin}(\mathsf{AP})\quad\Leftrightarrow\quad(u,x,y)\in\mathrm{Argmin}(\mathsf{P}) \tag{4.4}$$

provided $y=F(t,x,u)$.

Proof. Note that $x\in W^{1,p_1}(0,T;\mathbb{R}^n)\subset L^\infty(0,T;\mathbb{R}^n)$ and $u\in L^p(0,T;\mathbb{R}^m)$ implies $y\in L^{p_2}(0,T;\mathbb{R}^n)$ provided $y=F(t,x,u)$ and (4.2) is assumed. The condition (4.1) then ensures that the triple (u,x,y) is admissible for (P) if and only if the couple (u,x) is admissible for (AP). Then, if h is taken as in (4.3), the problems (P) and (AP) minimize the same functional essentially on the same admissible domain. □

Let us assume, for simplicity, $F(t,\cdot,u)$ uniformly Lipschitz-continuous as expressed by (4.5f), so that, in view of (3.1), it is natural here to impose the following basic data qualification concerning growth, certain Lipschitz continuity, and coercivity:

$$\bigl|f(t,x,y,u)\bigr| \leqslant \bigl(a_{p_1}(t)+c|u|^{p/p_1}\bigr)\bigl(1+|x|+|y|\bigr), \tag{4.5a}$$

$$\begin{aligned}&\bigl|f(t,x,y,u)-f(t,\tilde{x},\tilde{y},u)\bigr|\\ &\quad\leqslant \bigl(a_1(t)+\beta\bigl(|x|+|\tilde{x}|\bigr)+c|u|^p\bigr)\bigl(|x-\tilde{x}|+|y-\tilde{y}|\bigr),\end{aligned} \tag{4.5b}$$

$$\varepsilon|u|^p \leqslant h(t,x,y,u) \leqslant a_1(t)+\beta(|x|)+c|y|^{p_2/p_1}+c|u|^p, \tag{4.5c}$$

$$\begin{aligned}&\bigl|h(t,x,y,u)-h(t,\tilde{x},\tilde{y},u)\bigr|\\ &\quad\leqslant \bigl(a_1(t)+\beta\bigl(|x|+|\tilde{x}|\bigr)+c|u|^p\bigr)\bigl(|x-\tilde{x}|+|y-\tilde{y}|\bigr),\end{aligned} \tag{4.5d}$$

$$l(x) \geqslant 0, \tag{4.5e}$$

$$\bigl|F(t,x,u)-F(t,\tilde{x},u)\bigr| \leqslant \ell_0|x-\tilde{x}|, \tag{4.5f}$$

where $x,\tilde{x}\in\mathbb{R}^{n_1}$, $y,\tilde{y}\in\mathbb{R}^{n_2}$, $u\in\mathbb{R}^m$, $a_1\in L^1(0,T)$, $a_{p_1}\in L^{p_1}(0,T)$, $c\in\mathbb{R}$, $\varepsilon>0$, and $\beta:\mathbb{R}^+\to\mathbb{R}^+$ arbitrary continuous increasing. Let us only remark that (4.5f) could be weakened by, e.g., letting ℓ_0 depend on t and u provided the other conditions (4.5a)–(4.5d) would be made stronger; for simplicity, we did not treat such level of generality. Let us further define the manifold $\mathcal{M}(t,x)\subset\mathbb{R}^{n_2}\times\mathbb{R}^m$ by

$$\mathcal{M}(t,x) := \bigl\{(y,u)\in\mathbb{R}^{n_2}\times U(t);\ g(t,x,y,u)=0\bigr\} \tag{4.6}$$

and the orientor field $Q_{\mathcal{M}}(t,x)\subset\mathbb{R}\times\mathbb{R}^{n_1}$ by

$$\begin{aligned}Q_{\mathcal{M}}(t,x) := \bigl\{&(a,b)\in\mathbb{R}\times\mathbb{R}^{n_1};\ a\geqslant h(t,x,y,u),\\ &b=f(t,x,y,u),\ (y,u)\in\mathcal{M}(t,x)\bigr\}\end{aligned} \tag{4.7}$$

appearing naturally in the following existence result:

Proposition 1. *Let* (4.1), (4.2), *and* (4.5) *be valid and* $Q_{\mathcal{M}}(t,x)$ *be convex for all* $x\in\mathbb{R}^{n_1}$ *and a.a.* $t\in(0,T)$. *Then* (P) *has a solution.*

Proof. Again we will employ the transformation (4.3). Then (3.3) results, after the substitution $n_1:=n$, $\varphi:=f\circ F$, $\psi:=h\circ F$ as in (4.3), and $y:=F(t,x,u)$, to

$$\begin{aligned}Q(t,x) :=& \bigl\{(a,b)\in\mathbb{R}\times\mathbb{R}^{n_1};\ a\geqslant h\bigl(t,x,F(t,x,u),u\bigr),\\ &\quad b=f\bigl(t,x,F(t,x,u),u\bigr),\ u\in U(t)\bigr\}\\ =& \bigl\{(a,b)\in\mathbb{R}\times\mathbb{R}^{n_1};\ a\geqslant h(t,x,y,u),\\ &\quad b=f(t,x,y,u),\ y:=F(t,x,u),\ u\in U(t)\bigr\}\\ =& \bigl\{(a,b)\in\mathbb{R}\times\mathbb{R}^{n_1};\ a\geqslant h(t,x,y,u),\\ &\quad b=f(t,x,y,u),\ (y,u)\in\mathcal{M}(t,x)\bigr\}\\ =:& Q_{\mathcal{M}}(t,x).\end{aligned}$$

Then, the assumed convexity of $Q_{\mathcal{M}}(t,x)$ results to the convexity of Q, so that, by Theorem 1, the auxiliary problem (AP) has a solution (u,x). Using Lemma 1 (the implication $\Rightarrow$ in (4.4)) and putting $y = F(t,x,u)$, we get a solution to (P). $\square$

For the purpose of the optimality conditions, we will need also certain smoothness of f, g, h and l with respect to x and y:

$$\max\left(\left|\frac{\partial f}{\partial x}(t,x,y,u)\right|, \left|\frac{\partial f}{\partial y}(t,x,y,u)\right|\right)$$
$$\leqslant a_{p_1} + \beta(|x|) + c|y|^{p_2/p_1} + c|u|^{p/p_1}, \tag{4.8a}$$
$$\left|\frac{\partial f}{\partial x}(t,x,y,u) - \frac{\partial f}{\partial x}(t,\tilde{x},y,u)\right|$$
$$\leqslant \big(a_{q_1}(t) + \beta\big(|x| + |\tilde{x}|\big) + c|u|^{p/p_1}\big)|x-\tilde{x}|, \tag{4.8b}$$
$$\left|\frac{\partial f}{\partial y}(t,x,y,u) - \frac{\partial f}{\partial y}(t,\tilde{x},\tilde{y},u)\right|$$
$$\leqslant \big(a_{q_1}(t) + \beta\big(|x| + |\tilde{x}|\big) + c|u|^{p/p_1}\big)\big(|x-\tilde{x}| + |y-\tilde{y}|\big), \tag{4.8c}$$
$$\max\left(\left|\frac{\partial h}{\partial x}(t,x,y,u)\right|, \left|\frac{\partial h}{\partial y}(t,x,y,u)\right|\right)$$
$$\leqslant a_1(t) + \beta(|x|) + c|y|^{p_2} + c|u|^{p}, \tag{4.8d}$$
$$\left|\frac{\partial h}{\partial x}(t,x,y,u) - \frac{\partial h}{\partial x}(t,\tilde{x},y,u)\right|$$
$$\leqslant \big(a_1(t) + \beta\big(|x| + |\tilde{x}|\big) + c|u|^{p}\big)|x-\tilde{x}|, \tag{4.8e}$$
$$\left|\frac{\partial h}{\partial y}(t,x,y,u) - \frac{\partial h}{\partial y}(t,\tilde{x},\tilde{y},u)\right|$$
$$\leqslant \big(a_1(t) + \beta\big(|x| + |\tilde{x}|\big) + c|u|^{p}\big)\big(|x-\tilde{x}| + |y-\tilde{y}|\big), \tag{4.8f}$$

$\dfrac{\partial g}{\partial y}(t,x,y,u)$ is a regular $(n_2 \times n_2)$-matrix and

$$\left[\frac{\partial g}{\partial y}\right]^{-1}(t,x,y,u) \leqslant \beta(|x|), \tag{4.8g}$$

l continuously differentiable, (4.8h)

$$\left|\frac{\partial F}{\partial x}(t,x,u) - \frac{\partial F}{\partial x}(t,\tilde{x},u)\right| \leqslant \ell_1|x-\tilde{x}|. \tag{4.8i}$$

Let us remark that (4.8g) ensures the existence of the implicit function F only locally through the well-known implicit-function theorem so it cannot imply the condition (4.1) and, conversely, (4.1) does not imply (4.8g) even if g is smooth as seen simply for the example $g(t,x,y,u) = y^3 - u^3$.

Proposition 2. *Let* (4.1), (4.2), (4.5), *and* (4.8) *be valid and* (u, x, y) *solves* (P). *Then the maximum principle*

$$\mathcal{H}_{x,\lambda}\big(t, y(t), u(t)\big) = \max_{(\tilde{y},\tilde{u})\in\mathcal{M}(t,x(t))} \mathcal{H}_{x,\lambda}(t, \tilde{y}, \tilde{u}), \tag{4.9}$$

where the manifold $\mathcal{M}$ *is from* (4.6) *and the Hamiltonian* $\mathcal{H}_{x,\lambda} : (0, T) \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}$ *is defined by*

$$\mathcal{H}_{x,\lambda}(t, y, u) := \lambda(t)^{\mathrm{T}} \cdot f\big(t, x(t), y, u\big) - h\big(t, x(t), y, u\big) \tag{4.10}$$

with $\lambda \in W^{1,1}(0, T; \mathbb{R}^{n_1})$ *solving, together with* $\mu \in L^1(0, T; \mathbb{R}^{n_2})$, *the adjoint DAEs*

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\partial h}{\partial x}(t, x, y, u)^{\mathrm{T}} - \frac{\partial f}{\partial x}(t, x, y, u)^{\mathrm{T}}\lambda - \frac{\partial g}{\partial x}(t, x, y, u)^{\mathrm{T}}\mu,$$
$$\lambda(T) = \frac{\mathrm{d}l}{\mathrm{d}x}\big(x(T)\big), \tag{4.11a}$$

$$0 = \frac{\partial h}{\partial y}(t, x, y, u)^{\mathrm{T}} - \frac{\partial f}{\partial y}(t, x, y, u)^{\mathrm{T}}\lambda - \frac{\partial g}{\partial y}(t, x, y, u)^{\mathrm{T}}\mu. \tag{4.11b}$$

Proof. The assumption (4.1) enables us to transform the problem (P) into the form (AP) with φ and ψ given by (4.3). By Lemma 1 (the implication $\Leftarrow$ in (4.4)), (u, x) then solves (AP). Then the assumption (4.2) and (4.5) implies by a simple substitution just (3.1). Moreover, (4.2), (4.5f) and (4.8) imply (3.4); indeed, we have $|\partial F/\partial x| \leqslant \ell_0$ with ℓ_0 from (4.5f), and then by (4.8a) with (4.2) one gets

$$\begin{aligned}
&\left|\frac{\partial\varphi}{\partial x}(t, x, u)\right| \\
&\quad = \left|\frac{\partial f}{\partial x}\big(t, x, F(t, x, u), u\big) + \frac{\partial f}{\partial y}\big(t, x, F(t, x, u), u\big)\frac{\partial F}{\partial x}(t, x, u)\right| \\
&\quad \leqslant (1 + \ell_0)\big(a_{p_1} + \beta(|x|) + c|F(t, x, v)|^{p_2/p_1} + c|u|^{p/p_1}\big),
\end{aligned}$$

which gives (3.4a) with a suitable a_{p_1}, β, and c, and furthermore one gets

$$\begin{aligned}
&\left|\frac{\partial\varphi}{\partial x}(t, x, u) - \frac{\partial\varphi}{\partial x}(t, \tilde{x}, u)\right| \\
&\quad \leqslant \left|\frac{\partial f}{\partial x}\big(t, x, F(t, x, u), u\big) - \frac{\partial f}{\partial x}\big(t, \tilde{x}, F(t, \tilde{x}, u), u\big)\right| \\
&\qquad + \left|\frac{\partial f}{\partial y}\big(t, x, F(t, x, u), u\big) - \frac{\partial f}{\partial y}\big(t, \tilde{x}, F(t, \tilde{x}, u), u\big)\right|\left|\frac{\partial F}{\partial x}(t, x, u)\right| \\
&\qquad + \left|\frac{\partial f}{\partial y}\big(t, \tilde{x}, F(t, \tilde{x}, u), u\big)\right|\left|\frac{\partial F}{\partial x}(t, x, u) - \frac{\partial f}{\partial y}\big(t, \tilde{x}, F(t, \tilde{x}, u), u\big)\right|,
\end{aligned}$$

which gives (3.4b) if one uses (4.8b), (4.8c) and (4.8g) with (4.5f). The assumptions (3.4c) and (3.4d) can be obtained analogously by using (4.8d)–(4.8g). This allows us to use Theorem 2, so that (3.5)–(3.7) yields

$$\mathcal{H}^{\mathrm{aux}}_{x,\lambda}\big(t, u(t)\big) = \max_{\tilde{u}\in U(t)} \mathcal{H}^{\mathrm{aux}}_{x,\lambda}(t, \tilde{u}) \tag{4.12}$$

for a.a. $t \in (0, T)$, where the "auxiliary" Hamiltonian $\mathcal{H}^{\mathrm{aux}}_{x,\lambda} : (0, T) \times \mathbb{R}^m \to \mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{H}^{\mathrm{aux}}_{x,\lambda}(t, u) := \lambda(t)^{\mathrm{T}} \cdot f\big(t, x(t), F(t, x(t), u), u\big)&\\ - h\big(t, x(t), F\big(t, x(t), u\big), u\big),&\end{aligned} \tag{4.13}$$

where $x \in W^{1,p_1}(0, T; \mathbb{R}^{n_1})$ solves the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f\big(t, x, F(t, x, u), u\big), \quad x(0) = x_0, \tag{4.14}$$

and the adjoint state $\lambda \in W^{1,1}(0, T; \mathbb{R}^{n_1})$ satisfies the adjoint terminal-value problem

$$\begin{aligned}\frac{\mathrm{d}\lambda}{\mathrm{d}t} &= \frac{\partial\psi}{\partial x}^{\mathrm{T}} - \frac{\partial\varphi}{\partial x}^{\mathrm{T}}\lambda\\ &= \frac{\partial h}{\partial x}\big(t, x, F(t, x, u), u\big)^{\mathrm{T}} + \frac{\partial F}{\partial x}(t, x, u)^{\mathrm{T}}\frac{\partial h}{\partial y}\big(t, x, F(t, x, u), u\big)^{\mathrm{T}}\\ &\quad - \frac{\partial f}{\partial x}\big(t, x, F(t, x, u), u\big)^{\mathrm{T}}\lambda\\ &\quad - \frac{\partial F}{\partial x}(t, x, u)^{\mathrm{T}}\frac{\partial f}{\partial y}\big(t, x, F(t, x, u), u\big)^{\mathrm{T}}\lambda,\\ \lambda(T) &= \frac{\mathrm{d}l}{\mathrm{d}x}\big(x(T)\big).\end{aligned} \tag{4.15}$$

Then, in view of Lemma 1, $y = F(t, x, u)$. Using also the substitution $y := F(t, x(t), u)$ turns the Hamiltonian (4.13) into the form

$$\begin{aligned}&\lambda(t)^{\mathrm{T}} \cdot f\big(t, x(t), y, u\big) - h\big(t, x(t), y, u\big) =: \mathcal{H}_{x,\lambda}(t, y, u)\\ &\quad\text{for } (y, u) \in \mathcal{M}(t, x).\end{aligned} \tag{4.16}$$

Moreover, the maximum principle (4.12) then turns into (4.9), the initial-value problem (4.14) together with $y := F(t, x, u)$ and (4.1) gives just the DAEs in (1.1), and eventually (4.15) results to

$$\begin{aligned}\frac{\mathrm{d}\lambda}{\mathrm{d}t} &= \frac{\partial h}{\partial x}(t, x, y, u)^{\mathrm{T}} + \frac{\partial F}{\partial x}(t, x, u)^{\mathrm{T}}\frac{\partial h}{\partial y}(t, x, y, u)^{\mathrm{T}}n\\ &\quad - \frac{\partial f}{\partial x}(t, x, y, u)^{\mathrm{T}}\lambda - \frac{\partial F}{\partial x}(t, x, u)^{\mathrm{T}}\frac{\partial f}{\partial y}(t, x, y, u)^{\mathrm{T}}\lambda,\\ \lambda(T) &= \frac{\mathrm{d}l}{\mathrm{d}x}\big(x(T)\big).\end{aligned} \tag{4.17}$$

Due to (4.8g), there is μ solving (4.11b), namely

$$\mu := \left[\left(\frac{\partial g}{\partial y}\right)^{\mathrm{T}}\right]^{-1}\left(\left(\frac{\partial h}{\partial y}\right)^{\mathrm{T}} - \left(\frac{\partial f}{\partial y}\right)^{\mathrm{T}}\lambda\right).$$

Note that, by (4.8g), $[(\partial g/\partial y)^{\mathrm{T}}]^{-1}$ in $L^\infty(0,T;\mathbb{R}^{n_2\times n_2})$ and, by (4.8a) and (4.8e),

$$\left(\frac{\partial h}{\partial y}\right)^{\mathrm{T}} - \left(\frac{\partial f}{\partial y}\right)^{\mathrm{T}}\lambda \in L^1\big(0,T;\mathbb{R}^{n_2}\big)$$

so that certainly $\mu \in L^1(0,T;\mathbb{R}^{n_2})$, as claimed. By (4.1), $g(t,x,F(t,x,u),u) = 0$ so that

$$\frac{\partial g}{\partial x}(t,x,y,u) + \frac{\partial g}{\partial y}(t,x,y,u)\frac{\partial F}{\partial x}(t,x,u) = 0. \tag{4.18}$$

Multiplying (4.11b) by $\partial F/\partial x$ and using (4.18), one gets

$$\begin{aligned}&\frac{\partial h}{\partial y}(t,x,y,u)\frac{\partial F}{\partial x}(t,x,u) - \lambda^{\mathrm{T}}\frac{\partial f}{\partial y}(t,x,y,u)\frac{\partial F}{\partial x}(t,x,u)\\ &\quad = \mu^{\mathrm{T}}\frac{\partial g}{\partial y}(t,x,y,u)\frac{\partial F}{\partial x}(t,x,u) = -\mu^{\mathrm{T}}\frac{\partial g}{\partial x}(t,x,y,u).\end{aligned} \tag{4.19}$$

Substituting (4.19) into (4.17) gives (4.11a). This shows that (λ,μ) solves DAEs (4.11), as claimed. □

Remark 2. The maximum principle (4.9), (4.10) can be modified to a form more similar to (2.1), (2.2); namely

$$\mathcal{H}^1_{x,\lambda,\mu}\big(t,y(t),u(t)\big) = \max_{(\tilde{y},\tilde{u})\in\mathcal{M}(t,x(t))}\mathcal{H}^1_{x,\lambda,\mu}(t,\tilde{y},\tilde{u}), \tag{4.20}$$

where the Hamiltonian $\mathcal{H}^1_{x,\lambda,\mu} : (0,T)\times\mathbb{R}^{n_2}\times\mathbb{R}^m \to \mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{H}^1_{x,\lambda,\mu}(t,y,u) &:= \lambda(t)^{\mathrm{T}}\cdot f\big(t,x(t),y,u\big) + \mu(t)^{\mathrm{T}}\cdot g\big(t,x(t),y,u\big)\\ &\quad - h\big(t,x(t),y,u\big).\end{aligned} \tag{4.21}$$

However, note that the term $\mu\cdot g$ simply vanishes provided (y,u) and $(y(t),u(t))$ ranges the manifold $\mathcal{M}$.

Remark 3. If $U(t)$ and $h(t,x,y,\cdot)$ are convex and $[f,g](t,x,y,\cdot)$ affine, another modification is possible by introducing an auxiliary Hamiltonian, namely $h^F_{x,\lambda,\mu} := \mathcal{H}^1_{x,\lambda,\mu}\circ F : (0,T)\times\mathbb{R}^m \to \mathbb{R}$; i.e.,

$$h^F_{x,\lambda,\mu}(t,u) := \mathcal{H}^1_{x,\lambda,\mu}\big(t,F(t,x,u),u\big) \tag{4.22}$$

with F from (4.1). Note that (4.20) just says that $h^F_{x,\lambda,\mu}(t,\cdot)$ is maximized on the set $U(t)$ at the point $u(t)$. By smoothness and by (4.11b), it implies

$$
\begin{aligned}
N_{U(t)}\big(u(t)\big) &\ni \frac{\partial h^F_{x,\lambda,\mu}}{\partial u}(t,u)\\
&= \frac{\partial \mathcal{H}^1_{x,\lambda,\mu}}{\partial y}(t,y,u)\frac{\partial F}{\partial u}(t,x,u) + \frac{\partial \mathcal{H}^1_{x,\lambda,\mu}}{\partial u}(t,y,u)\\
&= \frac{\partial \mathcal{H}^1_{x,\lambda,\mu}}{\partial u}(t,y,u),
\end{aligned} \tag{4.23}
$$

where $N_U(u)$ denotes the normal cone to the set U at the point u; cf. also Chen and Hwang [23, Theorem 5.1] for $U = \mathbb{R}^m$ or De Pinho and Vinter [5, Theorem 3.2] for $U \subset \mathbb{R}^m$ possibly nonconvex. For a given $y = y(t)$, let us define the "reduced" Hamiltonian $\mathcal{H}^{i,\mathrm{red}}_{x,y,\lambda,\mu} : (0,T) \times \mathbb{R}^m \to \mathbb{R}$ by

$$
\mathcal{H}^{i,\mathrm{red}}_{x,y,\lambda,\mu}(t,u) := \mathcal{H}^i_{x,\lambda,\mu}\big(t, y(t), u\big); \tag{4.24}
$$

here $i = 1$. Then (4.23) implies that the function $\mathcal{H}^{1,\mathrm{red}}_{x,y\lambda,\mu}(t,\cdot)$, which is now assumed concave, is maximized over $U(t) \subset \mathbb{R}^m$, which is now assumed convex, at the point $u(t) \in \mathbb{R}^m$; i.e.,

$$
\mathcal{H}^{i,\mathrm{red}}_{x,\lambda,\mu}(t,y,u) = \max_{\tilde u \in U(t)} \mathcal{H}^{i,\mathrm{red}}_{x,\lambda,\mu}(t,y,\tilde u) \tag{4.25}
$$

for a.a. $t \in (0,T)$ and, here, for $i = 1$. This recovers, in less generality, the result by De Pinho and Vinter [5, Theorem 3.1] and explains why (2.4) cannot yield any counterexample if $\alpha \geqslant 0$. This also agrees with sensitivity analysis by Petzold et al. [24, Section 2.2]. However, it should be emphasized that selectivity of (4.25) can be lesser than selectivity of (4.20). For example, if $\mathcal{H}^{i,\mathrm{red}}_{x,y,\lambda,\mu}(t,\cdot)$ and $\mathcal{M}(t,x)$ are affine while $\mathcal{H}^i_{x,\lambda,\mu}(t,\cdot,\cdot)$ is strictly concave when restricted on $\mathcal{M}(t,x)$; then, if x, λ, and μ are given, (4.20) always picked up just one u while it can easily appear that $\mathcal{H}^{i,\mathrm{red}}_{x,y,\lambda,\mu}(t,\cdot)$ is constant and thus (4.25) looses selectivity completely at such time instance; cf. Remark 8 below.

Remark 4. Likewise (3.3), one can replace the convexity hypothesis on $Q_{\mathcal{M}}$ by the condition $\mathrm{co}([h \times f](t,x,\mathcal{M}(t,x))) \subset Q_{\mathcal{M}}(t,x)$, which can be, if used carefully in the proof of Proposition 1, combined with the maximum principle (4.9) by weakening it as

$$
\mathrm{co}\big([h \times f]\big(t,x,\mathcal{A}(t,x)\big)\big) \subset Q_{\mathcal{M}}(t,x) \tag{4.26}
$$

with an arbitrary estimate $\mathcal{A}(t,x)$ of the set of maximizers in (4.9); i.e.,

$$
\mathcal{A}(t,x) \supset \big\{(\tilde y,\tilde u) \in \mathcal{M}\big(t,x(t)\big);\ \mathcal{H}_{x,\lambda}(t,\tilde y,\tilde u) = \mathcal{H}_{x,\lambda}\big(t,y(t),u(t)\big)\big\}. \tag{4.27}
$$

This may sometimes enable us to get refined existence results even if the conventional orientor field (here $Q_{\mathcal{M}}$) is nonconvex; cf. [20–22] for analogous results in the case of ODEs or [16] for integral equations.

5. Index-2 problems

The condition (4.1) often cannot be fulfilled because the DAEs in question have a higher index k. We will demonstrate the needed modifications first for the case $k = 2$. We will assume g smooth, namely g of C^1-class, and

$$\frac{\partial g}{\partial y} \equiv 0 \quad \text{and} \quad \frac{\partial g}{\partial u} \equiv 0; \tag{5.1}$$

this means that g depends only on t and x. (The case $\partial g/\partial y \neq 0$ but singular would lead to various indices in particular equations, which would require a suitable combination of the presented results.) The latter condition in (5.1) implies the causality of DAEs. By differentiation of the algebraic equation (1.1b) in time and using also the differential equation (1.1a), one gets

$$0 = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial g}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} f, \tag{5.2}$$

where we used also (5.1) and omitted the arguments (t, x, y, u) for brevity. In analogy with (4.1), we will now assume the existence of an implicit function F corresponding to (5.2); i.e.,

$$\begin{aligned} &\exists F : (0, T) \times \mathbb{R}^{n_1} \times \mathbb{R}^m \to \mathbb{R}^{n_2}: \\ &\quad \left[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} f\right](t, x, y, u) = 0 \quad \Leftrightarrow \quad y = F(t, x, u). \end{aligned} \tag{5.3}$$

Moreover, it is also natural (and to some extent necessary) to assume x_0 compatible with the algebraic constraint (1.1b); i.e.,

$$g(0, x_0, \cdot, \cdot) = 0. \tag{5.4}$$

Lemma 2. *Let* (5.1), (5.3) *and* (5.4) *hold, and F from* (5.3) *fulfills* (4.2). *Then the equivalence* (4.4) *holds provided $y = F(t, x, u)$ and the auxiliary problem* (AP) *uses φ and ψ given again by* (4.3).

Proof. If $y := F(t, x, u)$, then by (5.1) and (5.3) we have always

$$0 = \left[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} f\right](t, x, y, u) = \frac{\mathrm{d}}{\mathrm{d}t} g(t, x, y, u) \tag{5.5}$$

from which we can deduce $g(t, x, y, u) = c_0$. Yet, $g(0, x(0), y(0), u(0)) = g(0, x_0, \cdot, \cdot) = 0$ by assumptions (5.1) and (5.4), concluding that the constant c_0 equals zero. Therefore, the triple (u, x, y) is admissible for (P) if and only if the couple (u, x) is admissible for (AP). Besides, $x \in W^{1,p_1}(0, T; \mathbb{R}^n) \subset L^\infty(0, T; \mathbb{R}^n)$ and $u \in L^p(0, T; \mathbb{R}^m)$ implies $y \in L^{p_2}(0, T; \mathbb{R}^n)$ provided $y = F(t, x, u)$ and (4.2) is assumed. So we get the same situation as in Lemma 1. □

We are basically in the same situation as in Section 4 but the data qualifications (4.5d) and (4.8i) now use F from (5.3) instead of (4.1), and the manifold $\mathcal{M}$ is defined now by

$$\mathcal{M}(t,x) := \left\{ (y,u) \in \mathbb{R}^{n_2} \times U(t);\ \left[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} f \right](t,x,y,u) = 0 \right\}. \tag{5.6}$$

Proposition 3. *Let* (5.1), (5.3), (5.4), (4.2), *and* (4.5) *with* F *from* (5.3) *be valid. Then*:

(i) *If* $Q_{\mathcal{M}}(t,x)$, *with* $\mathcal{M}$ *from* (5.6), *is convex for all* x *and a.a.* t, *then* (P) *has a solution.*

(ii) *Let, moreover,* (4.8a)–(4.8f), (4.8h) *and* (4.8i) *with* F *from* (5.3) *be valid and let*

$$\frac{\partial G}{\partial y} \textit{ be a regular } (n_2 \times n_2)\textit{-matrix},$$
$$\left[\frac{\partial G}{\partial y} \right]^{-1}(t,x,y,u) \leqslant \beta(|x|), \quad \textit{where} \quad G := \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} f. \tag{5.7}$$

Then any (u,x,y) *solving* (P) *satisfies the maximum principle* (4.9) *with* $\mathcal{M}$ *from* (5.6), $\mathcal{H}_{x,\lambda}$ *from* (4.10), *and* $\lambda \in W^{1,1}(0,T;\mathbb{R}^{n_1})$ *solving, together with* $\mu \in L^1(0,T;\mathbb{R}^{n_2})$, *the following adjoint DAEs*:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\partial h}{\partial x}(t,x,y,u)^{\mathrm{T}} - \frac{\partial f}{\partial x}(t,x,y,u)^{\mathrm{T}}\lambda - \frac{\partial G}{\partial x}(t,x,y,u)^{\mathrm{T}}\mu,$$
$$\lambda(T) = \frac{\mathrm{d}l}{\mathrm{d}x}\big(x(T)\big), \tag{5.8a}$$

$$0 = \frac{\partial h}{\partial y}(t,x,y,u)^{\mathrm{T}} - \frac{\partial f}{\partial y}(t,x,y,u)^{\mathrm{T}}\lambda - \frac{\partial G}{\partial y}(t,x,y,u)^{\mathrm{T}}\mu, \tag{5.8b}$$

with G *defined in* (5.7).

(iii) *If, in addition,* $U(t)$ *and* $h(t,x,y,\cdot)$ *are convex and* $[f,G](t,x,y,\cdot)$ *affine with* G *from* (5.7), *then any* (u,x,y) *solving* (P) *satisfies also the maximum principle* (4.25) *with* $i=2$ *and with the "reduced" Hamiltonian* $\mathcal{H}^{2,\mathrm{red}}_{x,y,\lambda,\mu}(t,u) := \lambda(t)^{\mathrm{T}} \cdot f(t,x(t),y(t),u) + \mu(t)^{\mathrm{T}} \cdot G(t,x(t),y(t),u) - h(t,x(t),y(t),u)$.

Proof. The point (i) is the same as Proposition 1 above. As to (ii), we can repeat the proof of Proposition 2 but with G defined in (5.7) in place of g; note that (5.7) now replaces (4.8g) which cannot be assumed due to (5.1). As to (iii), it just suffices to repeat the arguments of Remark 3. □

6. Index-3 problems

The derivation of F and $\mathcal{M}$ becomes complicated quite rapidly for increasing index k. Let us show it only for the index $k=3$ which also appears in nontrivial applications; cf. Section 7 below. Assuming f and g smooth, namely f of C^1-class and g of C^2-class, we now have to suppose, in addition to (5.1), also

$$\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\equiv 0 \quad \text{and} \quad \frac{\partial g}{\partial x}\frac{\partial f}{\partial u}\equiv 0. \tag{6.1}$$

The former condition means that the DAEs do not become a differential equation for the variable y and again, the latter condition implies the causality of DAEs. By differentiation of (5.2) in time and using also the differential equation (1.1a) and (6.1), one gets

$$\begin{aligned}0 &= \left(\frac{\partial^2 g}{\partial x\partial t}+\frac{\partial^2 g}{\partial x^2}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial^2 g}{\partial x\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial^2 g}{\partial x\partial u}\frac{\mathrm{d}u}{\mathrm{d}t}\right)f\\ &\quad+\frac{\partial g}{\partial x}\left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t}\right)\\ &\quad+\frac{\partial^2 g}{\partial t^2}+\frac{\partial^2 g}{\partial t\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial^2 g}{\partial t\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial^2 g}{\partial t\partial u}\frac{\mathrm{d}u}{\mathrm{d}t}\\ &= \frac{\partial^2 g}{\partial x^2}f^2+\frac{\partial g}{\partial x}\frac{\partial f}{\partial x}f+2\frac{\partial^2 g}{\partial x\partial t}f+\frac{\partial g}{\partial x}\frac{\partial f}{\partial t}+\frac{\partial^2 g}{\partial t^2}.\end{aligned} \tag{6.2}$$

Instead of (5.3), we now assume

$$\begin{aligned}&\exists F:(0,T)\times\mathbb{R}^{n_1}\times\mathbb{R}^m\to\mathbb{R}^{n_2}:\quad y=F(t,x,u)\quad\Leftrightarrow\\ &\quad\left[\frac{\partial^2 g}{\partial x^2}f^2+\frac{\partial g}{\partial x}\frac{\partial f}{\partial x}f+2\frac{\partial^2 g}{\partial x\partial t}f+\frac{\partial g}{\partial x}\frac{\partial f}{\partial t}+\frac{\partial^2 g}{\partial t^2}\right](t,x,y,u)=0.\end{aligned} \tag{6.3}$$

Again we are in the same situation as in Section 4 but the data qualifications (4.5d) and (4.8i) now use F from (6.3) instead of (4.1), and the manifold $\mathcal{M}$ is defined by

$$\begin{aligned}\mathcal{M}(t,x) := \Bigg\{(y,u)\in\mathbb{R}^{n_2}\times U(t);\ &\left[\frac{\partial^2 g}{\partial x^2}f^2+\frac{\partial g}{\partial x}\frac{\partial f}{\partial x}f\right.\\ &\left.+2\frac{\partial^2 g}{\partial x\partial t}f+\frac{\partial g}{\partial x}\frac{\partial f}{\partial t}+\frac{\partial^2 g}{\partial t^2}\right](t,x,y,u)=0\Bigg\}.\end{aligned} \tag{6.4}$$

Moreover, it is also natural (and to some extent necessary) to assume the initial velocity $(\mathrm{d}/\mathrm{d}t)x(0)$ compatible with the algebraic constraint (1.1b); i.e.,

$$\frac{\partial g}{\partial t}(0,x_0) = -\frac{\partial g}{\partial x}(0,x_0)\dot{x}_0, \tag{6.5a}$$

where

$$\dot{x}_0 := f(0,x_0,y,u),\qquad y\in\mathbb{R}^{n_2},\ u\in\mathbb{R}^m. \tag{6.5b}$$

Here we used the assumption (5.1) implying $(\partial/\partial t)g$ and $(\partial/\partial x)g$ independent of y and u. The important fact is that the right-hand side of (6.5a) does not depend on y and u because of the orthogonality (6.1). Though $\dot{x}_0$ itself may depend on y or u as indicated in (6.5b).

Lemma 3. *Let* (5.1), (5.4), (6.1), (6.3) *and* (6.5) *hold, and F from* (6.3) *fulfills* (4.2). *Then the equivalence* (4.4) *holds provided $y = F(t, x, u)$ and the auxiliary problem* (AP) *uses φ and ψ given again by* (4.3).

Proof. If $y := F(t, x, u)$, then by (5.1) and (5.3) we have always

$$0 = \left[\frac{\partial^2 g}{\partial x^2} f^2 + \frac{\partial g}{\partial x}\frac{\partial f}{\partial x} f + 2\frac{\partial^2 g}{\partial x \partial t} f + \frac{\partial g}{\partial x}\frac{\partial f}{\partial t} + \frac{\partial^2 g}{\partial t^2}\right](t, x, y, u)$$
$$= \frac{\mathrm{d}^2}{\mathrm{d}t^2} g(t, x, y, u), \tag{6.6}$$

from which we can deduce $g(t, x, y, u) = c_0 + c_1 t$. As in Lemma 2, (5.1) with (5.4) implies $c_0 = 0$. Moreover,

$$c_1 = \frac{\mathrm{d}}{\mathrm{d}t} g\big(0, x(0), y(0), u(0)\big) = \frac{\mathrm{d}}{\mathrm{d}t} g(0, x_0, y, u)$$
$$= \frac{\partial g}{\partial t}(0, x_0) - \frac{\partial g}{\partial x}(0, x_0) f(0, x_0, y, u) = 0 \tag{6.7}$$

due to (5.1) and (6.5); note that it holds independently of y and u. Therefore, the triple (u, x, y) is admissible for (P) if and only if the couple (u, x) is admissible for (AP), and $x \in W^{1,p_1}(0, T; \mathbb{R}^n)$ and $u \in L^p(0, T; \mathbb{R}^m)$ imply $y \in L^{p_2}(0, T; \mathbb{R}^n)$ provided $y = F(t, x, u)$ and (4.2) is assumed. So again we get the situation as in Lemma 1. □

Proposition 4. *Let* (5.1), (5.4), (6.1), (6.3), (6.5), (4.2) *and* (4.5) *with F from* (6.3) *be valid. Then*:

(i) *If $Q_{\mathcal{M}}(t, x)$, with $\mathcal{M}$ from* (6.4), *is convex for all x and a.a. t, then* (P) *has a solution.*

(ii) *Let, moreover,* (4.8a)–(4.8f), (4.8h) *and* (4.8i) *with F from* (6.3) *be valid and let*

$$\frac{\partial G}{\partial y} \textit{ be a regular } (n_2 \times n_2)\textit{-matrix}, \quad \left[\frac{\partial G}{\partial y}\right]^{-1}(t, x, y, u) \leqslant \beta(|r|),$$
$$\textit{where} \quad G := \frac{\partial^2 g}{\partial x^2} f^2 + \frac{\partial g}{\partial x}\frac{\partial f}{\partial x} f + 2\frac{\partial^2 g}{\partial x \partial t} f + \frac{\partial g}{\partial x}\frac{\partial f}{\partial t} + \frac{\partial^2 g}{\partial t^2}. \tag{6.8}$$

Then any (u, x, y) solving (P) *satisfies the maximum principle* (4.9) *with $\mathcal{M}$ from* (6.4), *$\mathcal{H}_{x,\lambda}$ from* (4.10), *and $\lambda \in W^{1,1}(0, T; \mathbb{R}^{n_1})$ solving, together with $\mu \in L^1(0, T; \mathbb{R}^{n_2})$, the adjoint DAEs* (5.8) *with G from* (6.8).

(iii) *If, in addition, $U(t)$ and $h(t,x,y,\cdot)$ are convex and $[f,G](t,x,y,\cdot)$ affine with G from* (6.8)*, then any (u,x,y) solving* (P) *satisfies also the maximum principle* (4.25) *with $i=3$ and and with the "reduced" Hamiltonian $\mathcal{H}^{3,\mathrm{red}}_{x,y,\lambda,\mu}(t,u) := \lambda(t)^{\mathrm T}\cdot f(t,x(t),y(t),u) + \mu(t)^{\mathrm T}\cdot G(t,x(t),y(t),u) - h(t,x(t),y(t),u)$ again with G from* (6.8)*.*

Proof. The same as of Proposition 3 above. □

Remark 5. In fact, the results from Sections 4–6 can be written in a unified way. Denoting by $i=1,\dots,3$ the index of the system, the manifold $\mathcal{M}$ entering the orientor field $Q_{\mathcal{M}}$ and the maximum principle (4.9), (4.10) has the form

$$\mathcal{M}(t,x) := \big\{(y,u)\in\mathbb{R}^{n_2}\times U(t);\ \big[\mathrm{D}^{i-1}g\big](t,x,y,u)=0\big\} \tag{6.9}$$

while the adjoint equations can be written in the form

$$\begin{aligned}&\frac{\mathrm d\lambda}{\mathrm dt} = \frac{\partial h}{\partial x}(t,x,y,u)^{\mathrm T} - \frac{\partial f}{\partial x}(t,x,y,u)^{\mathrm T}\lambda - \frac{\partial}{\partial x}\mathrm{D}^{i-1}g(t,x,y,u)^{\mathrm T}\mu,\\ &\quad\lambda(T) = \frac{\mathrm dl}{\mathrm dx}\big(x(T)\big),\end{aligned} \tag{6.10a}$$

$$0 = \frac{\partial h}{\partial y}(t,x,y,u)^{\mathrm T} - \frac{\partial f}{\partial y}(t,x,y,u)^{\mathrm T}\lambda - \frac{\partial}{\partial y}\mathrm{D}^{i-1}g(t,x,y,u)^{\mathrm T}\mu, \tag{6.10b}$$

where

$$\mathrm{D}^{i-1}g(t,x,y,u) := \frac{\mathrm d^{i-1}}{\mathrm dt^{i-1}}g(t,x,y,u)$$

with $(\mathrm d/\mathrm dt)x = f$ and $(\mathrm d/\mathrm dt)y = g$. Moreover, validity of the standard maximum principle (4.25) but with the reduced Hamiltonian

$$\begin{aligned}\mathcal{H}^{i,\mathrm{red}}_{x,y,\lambda,\mu}(t,u) &:= \lambda(t)^{\mathrm T}\cdot f\big(t,x(t),y(t),u\big)\\ &\qquad + \mu(t)^{\mathrm T}\cdot\mathrm{D}^{i-1}g\big(t,x(t),y(t),u\big) - h\big(t,x(t),y(t),u\big)\end{aligned} \tag{6.11}$$

is ensured provided $U(t)$ is convex and $\mathcal{H}^{i,\mathrm{red}}_{x,y,\lambda,\mu}(t,\cdot)$ concave. It is not difficult to see that all these results hold even for indices $i\geqslant 4$ provided sufficient smoothness of f and g and conditions like (5.1) and (6.1) are assumed.

7. Mechanical descriptor systems

We will briefly outline application of the results of Section 6 to an important class of index-3 DAEs arising in robotics. Let us consider the general formulation of equations of motion of multibody systems using redundant coordinates [8]. The *industrial robots* are just typical examples of multibody systems, the redundant coordinates being bounded by *holonomic* kinematic constraints.

Without loss of generality let us consider only the following *autonomous* case:

$$M(q)\frac{\mathrm{d}^2q}{\mathrm{d}t^2}+K\left(q,\frac{\mathrm{d}q}{\mathrm{d}t}\right)=J(q)^{\mathrm{T}}w+B(q,u),$$

$$q(0)=q_0,\quad \frac{\mathrm{d}q}{\mathrm{d}t}(0)=q_1, \tag{7.1a}$$

$$C(q)=0, \tag{7.1b}$$

where $q:(0,T)\to\mathbb{R}^n$ is a position (i.e., redundant, dependent or physical coordinates) of the robot, $M:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ a regular mass matrix, $K:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ involves Coriollis, centrifugal and possibly also friction forces, $C:\mathbb{R}^n\to\mathbb{R}^k$ describes kinematic constraints assumed smooth with $J:=(\mathrm{d}/\mathrm{d}q)C:\mathbb{R}^n\to\mathbb{R}^{k\times n}$ denoting the Jacobian matrix and $H:=(\mathrm{d}^2/\mathrm{d}q^2)C:\mathbb{R}^n\to\mathbb{R}^{k\times n\times n}$ (used later) denoting its Hessian, $w:[0,T]\to\mathbb{R}^k$ is the corresponding Lagrange multiplier expressing the reaction forces to these constraints, $u:[0,T]\to U\subset\mathbb{R}^m$ control variables as applied forces and $B:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ their transmission functions.

The task of an optimal control of the "robot" (7.1) is usually addressed as *optimal trajectory planning*. An often used cost functional is

$$I(q,u):=\int_0^T |q(t)-q_d(t)|^2+|u(t)|^2\,\mathrm{d}t, \tag{7.2}$$

where q_d is a desired trajectory. This expresses the requirements of a small difference from the desired trajectory competing with a small "energy" of the control. In view of the quadratic growth of I in terms of u, the natural choice is $p=2$ in (P).

Transformation to DAEs (1.1) can be performed by the following choice: $n_1:=2n$, $n_2:=k$,

$$x:=\left(q,\frac{\mathrm{d}q}{\mathrm{d}t}\right),\qquad y:=w, \tag{7.3a}$$

$$f(x,y,u)\equiv[f_1,f_2](x,y,u)$$

$$\text{with}\quad f_1(x,y,u):=x_2,$$

$$f_2(x,y,u):=M^{-1}(x_1)\big(J(x_1)^{\mathrm{T}}y+B(x_1,u)-K(x)\big), \tag{7.3b}$$

$$g(x,y,u):=C(x_1). \tag{7.3c}$$

We will show that these equations are of the index $k=3$ and identify simultaneously the manifold $\mathcal{M}=\mathcal{M}(x)$. Note that g does not involve the variable y so that (4.1) cannot apply. Both conditions in (5.1) are valid as C does not depend on y and u. Therefore, if g is differentiated once with respect to time, according to (5.3) one gets

$$\left[\frac{\partial g}{\partial t}+\frac{\partial g}{\partial x}f\right](t,x,y,u):=\left(\frac{\partial g}{\partial x_1},\frac{\partial g}{\partial x_2}\right)\cdot(f_1,f_2)$$
$$=\big(J(x_1),0\big)\cdot(x_2,f_2)=J(x_1)x_2=0. \qquad (7.4)$$

This constraint does not depend on the variable y, and therefore F from (5.3) cannot be found. However, the conditions in (6.1) are fulfilled due to the following orthogonality relations:

$$\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}=\left(\frac{\partial C}{\partial x_1},\frac{\partial C}{\partial x_2}\right)\cdot\left(\frac{\partial f_1}{\partial y},\frac{\partial f_2}{\partial y}\right)=(J,0)\cdot\big(0,M^{-1}J^{\mathrm T}\big)\equiv 0, \qquad (7.5a)$$
$$\frac{\partial g}{\partial x}\frac{\partial f}{\partial u}=\left(\frac{\partial C}{\partial x_1},\frac{\partial C}{\partial x_2}\right)\cdot\left(\frac{\partial f_1}{\partial u},\frac{\partial f_2}{\partial u}\right)=(J,0)\cdot\left(0,\frac{\partial B}{\partial u}\right)\equiv 0. \qquad (7.5b)$$

Therefore, if g differentiated twice with respect to time, according to (6.3) we get

$$G(t,x,y,u)$$
$$:=\left[\frac{\partial^2 g}{\partial x^2}f^2+\frac{\partial g}{\partial x}\frac{\partial f}{\partial x}f+2\frac{\partial^2 g}{\partial x\partial t}f+\frac{\partial g}{\partial x}\frac{\partial f}{\partial t}+\frac{\partial^2 g}{\partial t^2}\right](t,x,y,u)$$
$$=H(x_1)x_2^2+JM^{-1}(x_1)\big(J(x_1)^{\mathrm T}y+B(x_1,u)-K(x)\big)=0; \qquad (7.6)$$

of course, the term $Hx_2^2\equiv x_2^{\mathrm T}H(x_1)x_2\in\mathbb{R}^k$ means

$$\big[Hx_2^2\big]_\kappa=\sum_{i=1}^n\sum_{j=1}^n\left[\frac{\partial^2}{\partial x_{2,i}\partial x_{2,j}}C_\kappa\right]x_{2,i}x_{2,j}\quad\text{with }\kappa=1,\dots,k.$$

Now the variable y appears in this expression and, if $JM^{-1}J^{\mathrm T}$ is regular, we can express

$$y=F(x,u)=\big[JM^{-1}J^{\mathrm T}\big]^{-1}(x_1)$$
$$\times\big(J(x_1)M^{-1}(x_1)\big(K(x)-B(x_1,u)\big)-H(x_1)x_2^2\big). \qquad (7.7)$$

Assuming nondegeneracy of the holonomic constraints in the sense

$$\forall x_1\in\mathbb{R}^n:\quad \big|\det\big(JM^{-1}J^{\mathrm T}(x_1)\big)\big|\geqslant\varepsilon>0, \qquad (7.8)$$

the regularity (6.8) with G from (7.6) is valid, the assumptions (4.2), (4.5), (4.8a)–(4.8f), (4.8h) and (4.8i) are fulfilled if $|B(x_1,\cdot)|\leqslant C(1+|v|^{2/p_2})$ and $B(\cdot,v)$, as well as M^{-1}, J, and K are Lipschitz-continuous. Hence the manifold $\mathcal{M}$ from (6.4), now time-independent, can be explicitly obtained in the form

$$\mathcal{M}(x):=\big\{(y,u)\in\mathbb{R}^{k\times m}:\quad u\in U,$$
$$JM^{-1}(x_1)\big(K(x)-J(x_1)^{\mathrm T}y-B(x_1,u)\big)=H(x_1)x_2^2\big\}. \qquad (7.9)$$

This proves that the DAEs (7.1) are indeed of the index 3. The expression (7.7) of the implicit function F from (6.3) is the well-known elimination of the Lagrange multipliers without reduction of redundant coordinates into generalized ones [25]. The *compatibility* conditions (5.4) and (6.5) now read simply as

$$C(q_0)=0 \quad \text{and} \quad J(q_0)q_1=0; \tag{7.10}$$

i.e., the initial position q_0 of the robot fulfills the kinematic constraints while its initial velocity q_1 lies in the tangent space. Now, under the above data qualification, we can apply quite routinely Proposition 4 with several a posteriori modifications, e.g., omitting y- and u-independent terms in the Hamiltonian because such terms do not influence the maximum principles. This gives existence of (u,q,w) minimizing (7.2) under the constraint (7.1) and $u\in L^2(0,T;\mathbb{R}^m)$ provided

$$\big\{\big(a,\,B(x_1,u)\big);\ a\geqslant|u|^2,\ u\in U\big\} \text{ is convex for all } x_1. \tag{7.11}$$

Moreover, any such (u,q,w) satisfies the maximum principle

$$\mathcal{H}_{\lambda,q}\big(t,w(t),u(t)\big)=\max_{(\tilde w,\tilde u)\in\mathcal{M}(q(t),dq/dt)}\mathcal{H}_{\lambda,q}(t,\tilde w,\tilde u), \tag{7.12}$$

with the Hamiltonian $\mathcal{H}_{\lambda,q}:(0,T)\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ given by

$$\mathcal{H}_{\lambda,q}(t,y,u)=\lambda^{\mathrm{T}}(t)\cdot\Big(J\big(q(t)\big)^{\mathrm{T}}y+B\big(q(t),u\big)\Big)-|u|^2, \tag{7.13}$$

with $\lambda:=(M(q)^{\mathrm{T}})^{-1}\lambda_2$, where $\lambda_1,\lambda_2\in W^{1,1}(0,T;\mathbb{R}^n)$ and $\mu\in L^1(0,T;\mathbb{R}^k)$ solve the linear DAE adjoint to (7.1) written here, for simplicity, as the system

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t}=2(q-q_d)-\left[\frac{\partial f_2}{\partial x_1}\right]^{\mathrm{T}}\lambda_2-\left[\frac{\partial G}{\partial x_1}\right]^{\mathrm{T}}\left(q,\frac{\mathrm{d}q}{\mathrm{d}t},u\right)\mu,$$

$$\lambda_1(T)=0, \tag{7.14a}$$

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}t}=(\lambda_2+J\mu)^{\mathrm{T}}M^{-1}(q)\frac{\partial K}{\partial x_2}-\lambda_1-\frac{\mathrm{d}q}{\mathrm{d}t}^{\mathrm{T}}\big(H(q)+H^{\mathrm{T}}(q)\big)\mu,$$

$$\lambda_2(T)=0, \tag{7.14b}$$

$$0=(\lambda_2+J\mu)^{\mathrm{T}}M^{-1}J^{\mathrm{T}}(q). \tag{7.14c}$$

If U is convex and $B(r_1,\cdot)$ affine, which is a typical case of industrial robots, then (7.11) is trivially ensured and, moreover, one can derive the classical maximum principle (4.25) but here with the reduced Hamiltonian

$$\mathcal{H}^{3,\mathrm{red}}_{q,\lambda,\mu}(t,u)=\big(\lambda(t)+M^{-1}\,J(q)\mu(t)\big)^{\mathrm{T}}\cdot B\big(q(t),u\big)-|u|^2. \tag{7.15}$$

Remark 6. The quality of necessary optimality conditions relies on their sufficiency in some cases. Our maximum principle (7.12)–(7.14) is indeed also sufficient at least in case M is constant, C and K are affine, and $B(r_1,v):=B_1r_1+B_2(v)$, because the auxiliary problem (AP) is then linear/quadratic and (3.5)–(3.7) are then also sufficient optimality conditions. We conjecture that, as J from (7.2) is uniformly convex with respect to q, (7.12)–(7.14) remain sufficient even if the above affine assumptions are slightly perturbed; for such sort of investigations we refer, e.g., to Gabasov and Kirillova [26] in case of (AP) and to Vlček [27,28] in a certain special case of (7.1). Yet, real industrial robots may work in so strongly nonlinear regimes that sufficiency of the maximum principle cannot be expected.

Remark 7. The equivalence (4.4) is valid from the analytical point of view and could be applied just for the correct derivation of optimality conditions and the presented existence theory. The particular numerical solution would have to be done by the special algorithms for solving DAE, because the direct integration of (4.1), (5.3) or (6.3) leads otherwise into numerical instability.

Remark 8. Let us come back to the example (1.2a)–(1.2c) (cf. Fig. 1), and consider the optimal-control problem to minimize the interaction force $y := w$ in the connecting rod:

$$\begin{cases} \text{Minimize } \int_0^T y^2\,\mathrm{d}t, \\ \text{subject to } \frac{\mathrm{d}x}{\mathrm{d}t} = \left(x_2, \frac{u-y}{m_1}, x_4, \frac{y}{m_2}\right), \quad x(0) = (0, 0, l_{12}, 0), \\ \qquad\qquad 0 = x_3 - x_1 - l_{12} \text{ on } (0, T), \\ \qquad\qquad u(t) \in [-1, 1] \text{ for a.a. } t \in (0, T), \\ \qquad\qquad u \in L^\infty(0, T), \quad x \in W^{1,\infty}(0, T; \mathbb{R}^4), \\ \qquad\qquad y \in L^\infty(0, T). \end{cases} \tag{7.16}$$

Then the application of maximum principle from Proposition 4(ii) gives

$$\begin{aligned} &\mathcal{H}_{x,\lambda}\big(t, y(t), u(t)\big) \\ &\quad = \max_{(\tilde{y},\tilde{u}) \in \mathcal{M}(t,x(t))} \left(\lambda(t)^{\mathrm{T}} \cdot \left(x_2, \frac{\tilde{u} - \tilde{y}}{m_1}, x_4, \frac{\tilde{y}}{m_2}\right) - \tilde{y}^2\right) \\ &\quad = \max_{\tilde{u} \in U} \left(\lambda(t)^{\mathrm{T}} \cdot \left(x_2, \frac{\tilde{u}}{m_1 + m_2}, x_4, \frac{\tilde{u}}{m_1 + m_2}\right) - \left(\frac{m_2 \tilde{u}}{m_1 + m_2}\right)^2\right), \end{aligned} \tag{7.17}$$

where $\mathcal{M}(t, x(t)) = \{(y, u) \in \mathbb{R} \times U;\ y/m_2 = (u - y)/m_1\}$. On the other hand, the maximum principle (4.25) from Proposition 4(iii) maximizes the reduced Hamiltonian

$$\begin{aligned} \mathcal{H}^{3,\mathrm{red}}_{x,y,\lambda,\mu}(t, u) = {} & \lambda(t)^{\mathrm{T}} \cdot \left(x_2, \frac{u - y(t)}{m_1}, x_4, \frac{y(t)}{m_2}\right) \\ & + \mu(t)^{\mathrm{T}} \cdot \left(\frac{y(t)}{m_2} - \frac{u - y(t)}{m_1}\right) - y^2(t) \end{aligned} \tag{7.18}$$

over U. The algebraic part of the adjoint system (5.8b) with (6.8) yields

$$\mu = \left(\frac{\lambda_2}{m_1} - \frac{\lambda_4}{m_2} + 2y\right) \frac{m_1 m_2}{m_1 + m_2}, \tag{7.19}$$

and the detailed information extracted from (7.17) gives

$$u = (\lambda_2 + \lambda_4)\frac{m_1 + m_2}{2m_2^2} \tag{7.20}$$

provided u is in the interior of U. Furthermore, $y = F(t, x, u) = m_2 u/(m_1 + m_2)$, and substituting (7.20) into it and then into (7.19) eventually gives $\mu = \lambda_2$. Using it in (7.18) puts u completely off, and therefore (7.18) does not yield any information if $u \in \mathrm{int}(U)$.

Let us still remark that the approach by Devdariani and Ledyaev [6, Corollary on pp. 82–83] cannot be applied to (7.1) at all, because [6, Hypothesis C] does not fit with the holonomic constraint (7.1b).

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